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OPTIMAL REPLACEMENT OF CONSUMER DURABLES

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1. Introduction

The problem of determining the optimal procedure for the replacement of capital or military equipment has been a constant concern of industrial firms and military organizations. Numerous studies have been made in the field from early days.

Following the pioneer work of J.S. Taylor [24] and Harold Hotelling [14], Gabriel Preinreich [21] introduces the important concept of the replacement chain, which shows that the economic replacement of a machine is affected by the entire chain of successive renewals over the firm's planning horizon. Armen Alchian [2] considers the replacement problem in the explicit functional framework and shows the simple algorithm of the solution. Dynamic programming technique is successfully applied by Richard Bellman [3], and further extended by Stuart Dreyfus [6]. Dreyfus presents a general solution to the problem of evaluating the decision to keep or to replace involving a finite, fixed time horizon model.

Conceived and developed in conjunction with the investment and inventory theory, the study of optimal replacement has never really extended its benefits to the study of consumer durables.[1]

The primary objective of this paper is to study the replacement problems within the inter-temporal utility maximization framework.[2] A consumer has a finite, fixed time horizon (T)[3] and derives utility from the consumption of both durables and non-durables. In the paper we choose housing as an example of a consumer durable.

The paper is organized as follows: Section 2 presents a short survey of the literature of replacement theory. Section 3 introduces a fairly detailed exposition of the model, in which replacement occurs once. Section 4 derives and characterizes the optimal consumption sequence. The well-known Fisherian result is observed, i.e., the optimal consumption sequence is increasing (decreasing) provided the ratio of interest rate to the subjective time preference is larger (smaller) than unity. Section 5 considers the problem of comparative dynamics, i.e., changing some parameters such as the price of a new house or the market rate of interest and observing the corresponding changes in the consumption sequence. It is noted that the change in the price of a new house shifts the whole consumption sequence either upward or downward depending upon what happens to the final stock of wealth as the price changes. The ambiguity of the results presented arises because of the presence of "wealth effects" which correspond closely with comparable "income effects" in static analysis. The changes in the market rate of interest have a less simple effect upon the consumption sequence than the price change. The effect of price change and interest rate change upon the optimal replacement time is also examined in Section 5.

2. A Survey of the Literature in the Replacement Theory

The theory of replacement has been founded upon the objective of spacing equipment replacements so as to maximize the "goodwill," or the present value of quasi-rent stream less the present value of all capital outlays, associated with a chain of successive equipment renewals in the framework of an entrepreneur's investment horizon.[4] Depending upon the nature of the horizon we may separate the replacement problem into the following three categories: (1) finite chain, (2) infinite chain, (3) fixed time horizon.

(1) Finite Chain. The entrepreneur's horizon extends over a finite chain of machines, i.e., he plans the introduction of a machine which is

to be replaced some given finite number of times in the future. The problem is stated as one of maximizing an expression of the form (1) below with respect to ℓ_j $(j = 1, 2, \cdots, n)$, the period of the respective machine's service.

$$(1) \quad G = \left\{ \int_o^{L_1} Q(0,t) e^{-\int_o^t r(\tau)d\tau} dt + S_1(L_1, C_1) e^{-\int_o^{L_1} r(\tau)d\tau} - C_1 \right\}$$

$$+ \left\{ \int_{L_1}^{L_2} Q(L_1, t - L_1) e^{-\int_o^t r(\tau)d\tau} dt + S_2(L_2, C_2) e^{-\int_o^{L_2} r(\tau)d\tau} - C_2 e^{-\int_o^{L_1} r(\tau)d\tau} \right\}$$

$$+ \cdots + \left\{ \int_{L_{n-1}}^{L_n} Q(L_{n-1}, t - L_{n-1}) e^{-\int_o^t r(\tau)d\tau} dt + S_r(L_n, C_n) e^{-\int_o^{L_n} r(\tau)d\tau} \right.$$

$$\left. - C_n e^{-\int_o^{L_{n-1}} r(\tau)d\tau} \right\}$$

To simplify the notation we adopt L_k such that $L_k = \Sigma_{j=1}^{k} \ell_j \quad k = 1, 2, \cdots, n.$

G	- the good will of a finite chain (n) of machines
$Q(L_j, a)$[5]	- the quasi-rent of the machine purchased at time L_j of age a
$S_j(L_j, C_j)$	- the scrap value of the jth machine at time L_j
C_j	- the cost of the jth machine
$r(\tau)$	- the rate of interest.

For brevity we assume the entrepreneur makes the replacement with the identical type of machine and we also assume the constancy of the scrap value and the rate of interest. Thus we have the following "goodwill" function.

$$(2) \quad G = \left\{ \int_o^{L_1} Q(t)e^{-rt}dt + Se^{-rL_1} - C \right\} + \left\{ \int_{L_1}^{L_2} Q(t - L_1)e^{-rt}dt + Se^{-rL_2} \right.$$

$$\left. - Ce^{-rL_1} \right\} + \cdots + \left\{ \int_{L_{n-1}}^{L_n} Q(t - L_{n-1})e^{-rt}dt + Se^{-rL_n} - Ce^{-rL_{n-1}} \right\}$$

Following Preinreich, instead of differentiating the above expression with respect to ℓ_j $(j = 1, \cdots, n)$ we differentiate the conceptionally identical but operationally easier form below with respect to T_j $(j = 1, 2, \cdots, n)$.

$$(3) \quad G_j = \int_o^{T_j} Q(t)e^{-rt}dt + (S + G_{j-1})e^{-rT_j} - C$$

Here the subscript $j = 1, 2, \cdots, n$ indicates the number of links in the chain from the end of the horizon. Thus T_1 corresponds to ℓ_n and T_n corresponds to ℓ_1 above.

Assuming the second order condition is satisfied, the maximizing condition is $dG_j/dT_j = 0$, i.e.,

$$(4) \quad Q(T_j) = r(S + G_{j-1})$$ [6]

Beginning with $G_o = 0$, the above equation determines T_1, the economic life of the last machine in the chain. Substituting this into (3) yields G_1, which is in turn inserted into (4) to find T_2. We continue this operation to find T_j and G_j for $j = 1, 2, \cdots, n$.[7] The maximizing condition (4) states that the quasi-rent at the time of discarding must be equal to the interest on the scrap value plus interest on all future "goodwills."

Given the quasi-rent declines as the machine ages, we observe each machine will have a longer optimum life than its predecessor in the chronological chain.[8] The "goodwill," or the present value of quasi-rent less all the capital outlays increases as the number of replacements increase.[9]

One special case of this finite chain is when the entrepreneur considers a machine isolated from the future course of events after its scrapping, in other words he takes no account of the machine that will replace it, i.e., $n = 1$. We have the quasi-rent function in the usual form below.

(5) $\quad Q(t) = zx(t) - M(t).$

z - known market price of the product

$x(t)$ - rate of production

$M(t)$ - combined rate of all expenses except depreciation and interest.

Substituting (5) into (3) and (4) we obtain

(6) $$G = \int_0^T \{zx(t) - M(t)\} e^{-rt} dt + Se^{-rT} - C$$

(7) $\quad zx(T) - M(T) = rS.$

The work of J.S. Taylor [24], later simplified and refined by H. Hotelling [14] comes into this category with some modification.[10] Instead of maximizing the "good will" they try to minimize "unit cost" w of the form below.[11]

(8) $$w = \frac{\int_0^T M(t)e^{-rt}dt - Se^{-rt} + C}{\int_0^T x(t)e^{-rt}dt}$$

Differentiate the above equation with respect to T and set it equal to zero, we obtain the minimizing condition of the form

(9) $\quad w(T)x(T) - M(T) = rS.$

It is clear that the respective optimal life of a machine arrived at in the above two methods, i.e., the solutions to equation (7) and equation (9), are quite different in general.

2. <u>Inifinite Chain.</u> If the entrepreneur's horizon extends over an infinite chain of machines, then the future "goodwill" at each period must be the same due to the nature of infinity. Thus we are to maximize the expression of the form below in each period with respect to T_j $\quad (j = 1, 2, \cdots)$.

(10) $$G = \int_0^{T_j} Q(t)e^{-rt}dt + (S + G)e^{-rT_j} - C$$

And the maximizing condition

(11) $\quad Q(T_j) = r(S + G)$

Thus it is clear in the infinite chain case that each machine in the chain will have the same optimal life (T). Substituting (11) into (10) we obtain[12]

$$(12)\quad Q(T) = r\,[S + \frac{1}{1-e^{-rT}}\{\int_o^T Q(t)e^{-rt}dt + Se^{-rT} - C\}].$$

which states that the quasi-rent at the time of replacement must be equal to the interest on its scrap value plus interest on all future "goodwills," which is of course constant.

3. Fixed Time Horizon. "Some types of services may not be needed for an indefinitely long period, and some agencies may view themselves as being appropriately concerned only with the truncated future, perhaps only with the next 20 or 30 years."[13] If the entrepreneur's time horizon is thus fixed, then what will be the optimal replacement policy? Should he replace the machine at all? If he replaces, how often and when?

The information from the analysis of the finite chain will partially help us in answering the number of replacements to be made but it does not answer the question of when to replace. Let $\overline{T}$ be the fixed time horizon, then if $\overline{T}$ is equal or greater than the optimal horizon associated with some fixed number of machines, $\overline{n}$, then the optimal number of machines required is greater than or equal to $\overline{n}$,[14] i.e.,

$$(13)\qquad \overline{T} \geq T(\overline{n}) \Longrightarrow n(\overline{T}) \geq \overline{n}$$ [15]

where $T(\overline{n})$ - optimal time horizon associated with $\overline{n}$ number of machines

$$T(\overline{n}) = \Sigma_{i=1}^{\overline{n}} T_i\,;$$

$n(\overline{T})$ - optimal number of machines associated with the time horizon $\overline{T}$.

Suppose an optimal number of replacements $n(\overline{T})$ is chosen[16] given fixed horizon $\overline{T}$, then the optimal life length of each machine (m_j) can be obtained by differentiating the form below with respect to m_j $\quad j = 1, \cdots, n(\overline{T})$ and setting them equal to zero.

$$(14)\qquad G = \sum_{i=o}^{n(\overline{T})-1}\{\int_{M_i}^{M_{i+1}} Q(t-M_i)e^{-rt}dt + Se^{-rM_{i+1}} - Ce^{-rM_i}\}$$

where $M_i = \Sigma_{j=1}^{i} m_j \quad i = 1, 2, \cdots, n(\overline{T});\quad M_o \equiv 0,\ M_{n(\overline{T})} \equiv \overline{T}$

$$(15)\qquad \frac{dG}{dm_j} = 0 \quad\Longrightarrow Q(m_j) = r[S + \{\int_o^{m_{j+1}} Q(t)e^{-rt} + \frac{Q(m_{j+1})}{r}e^{-rm_{j+1}} - C\}]$$

$$j = 1, 2, \cdots, n(\overline{T})-1.$$

The maximizing condition is somewhat similar to the condition (4) of the finite chain case, the quasi-rent of the jth machine at the time of replacement must be equal to the interest on the scrap value plus in the interest on the future "goodwill."

In contrast to the finite chain case, all we can say about the sequence of the optimal life of a machine $\left\{m_j\right\}_{j=1}^{n(\overline{T})}$ is that it takes on of the three forms below:[17]

(1) Constant sequence, i.e., $m_j = m\ \forall j$

(2) monotonically increasing sequence, i.e., $m_{j+1} > m_j\ \forall j$

(3) monotonically decreasing sequence, i.e., $m_{j+1} < m_j\ \forall j$.

R. Bellman [3] introduced a functional equation technique of dynamic programming to the theory of replacement concerning the infinite chain. He defines

(16) $f(\tau, t)$ = overall return obtained from a machine of age t at time τ, using an optimal replacement policy.

At each time τ, the entrepreneur chooses one of two alternatives. He may either keep (K) the machine for another time period, or he may purchase (P) a new machine. Thus, the functional equation for $f(\tau, t)$ is

$$(17) \quad f(\tau, t) = \text{Max} \begin{bmatrix} K: Q(\tau, t) + a f(\tau+1, t+1) \\ P: S(\tau, t) - C(\tau) + Q(\tau, 0) + a f(\tau+1, 1) \end{bmatrix}$$

where

- K - stands for the policy to keep the present machine
- P - stands for the policy to scrap the present machine and to purchase a new machine
- $Q(\tau, t)$ - quasi-rent of machine of age t at time τ
- a - a discount factor $a \in (0, 1)$
- $S(\tau, t)$ - scrap value of machine of age t at time τ
- $C(\tau)$ - the cost of purchasing a new machine at time τ.

For brevity assume the constancy of the scrap value and assume the quasi-rent and the cost of new machine do not depend upon the chronological year, then the indicator τ may be dropped from (17)

An optimal policy will have the form: keep a new machine until it is T years old and then purchase a new one, thus the following system of equations is derived

$$(18) \quad \begin{aligned} f(0) &= Q(0) + af(1) \\ f(1) &= Q(1) + af(2) \\ &\ \ \vdots \\ f(T-1) &= Q(T-1) + af(T) \\ f(T) &= S - C + Q(0) + af(1) \end{aligned}$$

Solving for f(0) recurrently, we obtain

$$(19) \quad f(0) = \frac{1}{1 - a^T} \sum_{t=o}^{T-1} Q(t)a^t + \frac{a^T}{1 - a^T} (S - C)$$

The entrepreneur chooses T such that it maximizes the right hand side of equation (19). Let us call this maximizing life length $\hat{T}$ and the associated return $f_{\hat{T}}(0)$ respectively. Then we have the following expression

$$(20) \quad f_{\hat{T}}(0) - f_{\hat{T}+1}(0) = \frac{a^{\hat{T}}(1 - a)}{(1 - a^{\hat{T}})(1 - a^{\hat{T}+1})} \left\{ \sum_{t=o}^{\hat{T}-1} Q(t)a^t + S - C \right\} - \frac{Q(\hat{T})a^{\hat{T}}}{1 - a^{\hat{T}+1}}$$

or

$$(21) \quad Q(\hat{T}) = - \left\{ f_{\hat{T}}(0) - f_{\hat{T}+1}(0) \right\} + \frac{1-a}{1 - a^{\hat{T}}} \left\{ \sum_{t=o}^{\hat{T}-1} Q(t)a^t + S - C \right\}$$

where $f_{\hat{T}+1}(0)$ is the return associated with the policy of keeping the machine for $\hat{T} + 1$ years.

Equation (21) is the discrete analogue to equation (12) above.[18]

The fixed time horizon case is studied by S. Dreyfus [6] in a manner similar to the above. He gives an explicit example where technological changes are taken into account.

3. The Model

Presently the consumer owns a house, but he plans to replace it by another some time in the future. He wants to determine the optimal sequence of saving and consumption, and the optimal timing of the replacement in his finite, fixed time horizon. He derives utility from the house in which he lives, from the rest of goods and services he consumes, and from the final stock of wealth. Thus the consumer's discounted present value of utility consists of the sum of his discounted instantaneous utility and the discounted utility of his final stock of wealth. We assume that the use value of a house can be approximated by the size and quality of the house and the environmental considerations such as the nature of the school district, tax rate, zoning, neighborhood, etc. We also assume that the use value of a house would decline over the years due to general wear and tear and obsolescence.

We use the following notations:

C_t - consumption of goods and services excluding housing at time t
h_t - the use value of house at time t
$U(C_t, h_t)$ - utility at time t
μ - rate of consumer's time preference
W_T - final stock of wealth
$V(W_T)$ - utility of the final stock of wealth
s - T-component vector of saving
t^* - replacement time, i.e., at time t^* he sells his old house and buys a new one
P - price of a new house
$\Psi(s, t^*, P)$ - discounted present value of utility.

Then the discounted present value of utility is given by

$$\Psi(s, t^*, P) = \sum_{t=1}^{T} U(C_t, h_t)(1+\mu)^{1-t} + V(W_T)(1+\mu)^{1-T} \tag{1.1}$$

where

$$h_t = \begin{cases} h^o e^{-\rho(t-t_o^*)} & t = 1, \cdots, t^* \\ h' e^{-\rho(t-t^*)} & t = t^*+1, \cdots, T \end{cases} \tag{1.1a}$$

h^o - use value of the old house at the time of its purchase $t_o^* \leq 0$
h' - use value of the new house at the time of its purchase $t^* \in [1, \cdots, T-1]$
ρ - depreciation rate of a house.

The consumer maximizes his discounted present value of utility (1.1) with respect to the saving in each period and the replacement time for the given price of a new house, subject to the following constraints:

$$C_t = y_t - s_t - d_t - m_t \qquad t = 1, \cdots, T \tag{1.2}$$

where
y_t - income at time t, exogenously given
s_t - saving at time t
d_t - mortgage payment at time t

$$d_t = \begin{cases} \bar{d} & t = 1, \cdots, t^* \\ f(P) & t = t^*+1, \cdots, T \end{cases} \tag{1.2a}$$

$\bar{d}$ - a given constant such that $\bar{d} > 0$ and f(P) is an increasing function of P such that $f(0) = 0$, $df/dP > 0$.[19]

m_t - maintenance cost of a house at time t, exogenously given

(1.2b) $$m_t = \begin{cases} m^o(t - t_o^*) & t = 1, \cdots, t^* \\ m'(t - t^*) & t = t^* + 1, \cdots, T \end{cases}$$

$m^i(\tau)$ - maintenance cost of ith house of age τ $i = 0, 1$

(1.3) $$S_t = \begin{cases} s_t + (1 + r)S_{t-1} & t \neq t^*, \ t = 1, \cdots, T \\ s_t + (1 + r)S_{t-1} + \{E^o_{t^*} - \phi(P)\} & t = t^* \end{cases}$$

where

S_t - stock of saving at time t

r - market rate of interest

$E^o_{t^*}$ - equity of his old house at time t^*

(1.3a) $$E^o_{t^*} = H^o_{t^*} - D^o_{t^*}$$

$H^o_{t^*}$ - market value of his old house at time t^* such that $H^o_{t^*}$ is a function of the use value of the house and the price of the new house, i.e.,

(1.3b) $$H^o_{t^*} = H(h^o_{t^*}, P)$$

$D^o_{t^*}$ - outstanding debt on his old house at time t^*

(1.3c) $$D^o_{t^*} = (1 + r)D^o_{t^*-1} - \bar{d}$$
$$= (1 + r)^{t^*} D^o_o - \bar{d} \sum_{t=1}^{t^*} (1 + r)^{t-1} \geq 0$$

$\phi(P)$ - downpayment on the new house such that

$$\frac{d\phi}{dP} = \phi'(P) \epsilon (0, 1), \ \phi(0) = 0.$$

Thus, given the initial stock of saving S_o and the debt D^o_o, we can express S_t as the following

(1.3') $$S_t = \begin{cases} (1+r)^t S_o + \sum_{k=1}^{t} (1 + r)^{t-k} s_k & t = 1, \cdots, t^* - 1 \\ (1 + r)^t S_o + \sum_{k=1}^{t} (1 + r)^{t-k} s_k + (1 + r)^{t-t^*} \{H^o_{t^*} - \phi(P)\} & t = t^*, \cdots, T \end{cases}$$

(1.4) $$W_T = S_T + H'_T - D'_T$$

where

W_T - final stock of wealth

S_T - final stock of saving

H'_T - market value of his new house at time T, assumed as a fraction of the cost per use value multiplied by the total use value at time T, i.e.,

(1.4a) $$H'_T = \gamma (P/h')h'_T, \quad \gamma \in (0, 1)$$

D'_T - outstanding debt on the new house at time T

(1.4b) $$D'_T = (1 + r)^{T-t^*} \{P - \phi(P)\} - f(P) \sum_{t=t^*+1}^{T} (1+r)^{t-t^*-1} \geq 0$$

or given $\bar{t}$,[20] time period allowed to complete the debt payment,

(1.4b') $$D'_T = \{P - \phi(P)\} B(t^*)$$

where

$$B(t^*) = \frac{1 - (1+r)^{T-(t^*-\bar{t})}}{1 - (1+r)^{-\bar{t}}} \in [0,1]$$

$$\frac{dB}{dt^*} \equiv B' > 0,$$

(1.5) $C_t \geq 0$ [21] $\forall t$

(1.6) $S_{t^*} \geq 0$

(1.7) $W_T \geq 0$

(1.8a) $U(C_t, h_t) > 0,\ V(W_T) > 0$ for $C_t, h_t > 0,\ W_T \geq 0$

(1.8b) $\frac{\partial}{\partial C_t} U(C_t, h_t) > 0,\ \frac{\partial}{\partial h_t} U(C_t, h_t) > 0,\ \frac{\partial}{\partial C_t} U(0, h_t) = \infty,$

$\frac{d}{dW_T} V(W_T) > 0,$ for $C_t, h_t > 0,\ W_T \geq 0.$

(1.8c) $\frac{\partial^2}{\partial C_t^2} U(C_t, h_t) < 0,\ \frac{d^2}{dW_T^2} V(W_T) < 0$ for $C_t, h_t < 0,\ W_T \leq 0.$

4. Derivation and Characterization of Optimal Consumption Sequence

The maximization can be performed in two steps:

(A) Maximize the discounted present value of utility with respect to s_t, $t = 1, \cdots, T$ for given P and t^*.

(B) Perform the above operation for every $t^* = 1, \cdots, T-1$ for the given P and choose t^* which maximizes the discounted present value of utility.

We note that the characterization of the optimal consumption sequence is invariant under different values of t^*,[22] hence it suffices to perform the maximization step (A) above to characterize the optimal consumption sequence.

We form the following Lagrange function:

$$\begin{aligned}(1.9)\quad L = &\sum_{t=1}^{T} U(y_t - s_t - d_t - m_t, h_t)(1+\mu)^{1-t} + (1+\mu)^{1-T} V[(1+r)^T S_o \\ &+ \sum_{k=1}^{T} (1+r)^{T-k} s_k + (1+r)^{T-t^*}\{E^o_{t^*} - \phi(P)\} + \gamma(P/h')h^1_T \\ &- \{(1+r)^{T-t^*}(P - \phi(P)) - f(P) \sum_{t=t^*+1}^{T} (1+r)^{t-t^*-1}\}] \\ &+ \alpha S_{t^*} + \beta W_T\end{aligned}$$

where α, β are the Lagrange multipliers associated with $S_{t^*} \geq 0,\ W_T \geq 0.$ The first order condition is given by

$$(1.10)\quad \frac{\partial U}{\partial C_t} \geq \frac{dV}{dW_T}\left(\frac{1+r}{1+\mu}\right)^{T-t} \quad \text{for } t = 1, 2, \cdots, T$$

(Equality holds for $S_{t^*}, W_T > 0$)

For the interior maximum case, i.e., $S_{t^*}, W_T > 0$, we derive from (1.8) and (1.10) above the following three sets of characterization depending upon the relationship between the housing consumption and the consumption of other goods and services in the utility function, i.e., whether their consumption is independent $\left(\frac{\partial^2 U}{\partial C_t \partial h_t} = 0\right)$, or non-complements $\left(\frac{\partial^2 U}{\partial C_t \partial h_t} \leq 0\right)$, or non-substitutes $\left(\frac{\partial^2 U}{\partial C_t \partial h_t} \geq 0\right)$.

Case a $\quad \frac{\partial^2 U}{\partial C_t \partial h_t} = 0$

The optimal consumption sequence $\{C_t\}_{t=1}^{T}$ is strictly increasing, constant, and strictly decreasing for $r > \mu$, $r = \mu$, $r < \mu$, respectively.

Case b $\quad \frac{\partial^2 U}{\partial C_t \partial h_t} \leq 0$

The optimal consumption sequences $\{C_t\}_{t=1}^{t^*}$, $\{C_t\}_{t=t^*+1}^{T}$ are both strictly increasing and non-decreasing for $r > \mu$, $r = \mu$, respectively.

Case c $\quad \frac{\partial^2 U}{\partial C_t \partial h_t} \geq 0$

The optimal consumption sequences $\{C_t\}_{t=1}^{t^*}$, $\{C_t\}_{t=t^*+1}^{T}$ are both non-increasing and strictly decreasing for $r = \mu$, $r < \mu$, respectively.

We note in all of the three cases, the importance of the relative magnitude of the interest rate to the rate of subjective time preference in determining the optimal consumption sequence. In general if the interest rate is greater than the rate of time preference, it is preferable to save more now and earn the interest which can be spent later without undue sacrifice. The converse also holds.

5. Comparative Dynamics

<5.1> First we consider the effect of change in the price of a new house P on the optimal consumption sequence $\{C_t\}_{t=1}^{T}$ and the optimal replacement timing t^*:

From (1.10) above we derive the following

$$\text{(1.11)} \qquad \frac{\partial U/\partial C_j}{\partial U/\partial C_i} = \left(\frac{1+r}{1+\mu}\right)^{i-j} \qquad \text{for } S_{t^*}, W_T > 0 \text{ and } i, j = 1, \cdots, T.$$

We differentiate (1.11) with respect to P to obtain

$$\text{(1.12)} \qquad \frac{dC_j/dP}{dC_i/dP} = \left(\frac{\partial^2 U/\partial C_i^2}{\partial^2 U/\partial C_j^2}\right)\left(\frac{1+r}{1+\mu}\right)^{i-j} > 0$$

$$\text{for } S_{t^*} > 0, W_T > 0 \text{ and } i, j = 1, \cdots, T.$$

which states that the price change will shift the whole consumption sequence. Thus if we know what happens to the level of consumption in any one period, we know for the rest of periods. Proposition 1 and its corollary show us if we know what happens to the final stock of wealth as the price changes, then we can tell the corresponding changes in the optimal consumption sequence and the present value of utility.

Proposition 1

For given t^* assume $S_{t^*}, W_T > 0$, then $dW_T/dP \geq 0$ implies[23]

(i) $dC_t/dP \geq 0 \ \forall t$, (ii) $d\Psi/dP > 0$ (all equality holds for $dW_T/dP = 0$)

where $\Psi(s, t^*, P) = \sum_{t=1}^{T} U(T_t, h_t)(1+\mu)^{1-t} + V(W_T)(1+\mu)^{1-T}$.

Proof: (i) To show $dC_t/dP \geq 0 \quad \forall\ t=1,2,\cdots,T$.

$dW_T/dP \geq 0$ implies $\frac{d}{dP} V'(W_T) \leq 0$ by strict concavity of $V(W_T)$. From (1.10) $\frac{\partial U}{\partial C_t} = V'\left(\frac{1+r}{1+\mu}\right)^{T-t} \forall t$, thus $\frac{d}{dP}\left(\frac{\partial U}{\partial C_t}\right) \leq 0 \quad \forall t$.

However $\frac{dh_t}{dP} = 0$ and $\frac{\partial^2 U}{\partial C_2^t} < 0$ imply $\frac{dC_t}{dP} \geq 0 \quad \forall t$.

Q. E. D.

(ii) To show $d\Psi/dP \geq 0$

Differentiate (1.1) with respect to P to obtain

$$\frac{d\Psi}{dP} = \sum_{t=1}^{T} \frac{\partial U}{\partial C_t} \frac{dC_t}{dP} \left(1+\mu\right)^{1-t} + \frac{dV}{dW_t} \frac{dW_T}{dP}\left(1+\mu\right)^{1-T} \tag{1.13}$$

By the result above $dC_t/dP \geq 0 \ \forall t$ and by hypothesis $dW_T/dP \geq 0$, we have $d\Psi/dP \geq 0$.

Q. E. D.

Corollary 1

For given t^* assume $S_{t^*}, W_T > 0$, then $dW_T/dP < 0$ implies[24]

(i) $dC_t/dP < 0 \quad \forall t$, (ii) $d\Psi/dP < 0$.

In general, it is not possible to find the directional change of the optimal replacement time (t^*) as the price of a new house changes. The change of the replacement time in either direction would have both a positive and negative effect on the utility. However, by having some additional

restrictions on the nature of the utility functions and the end conditions, we obtain the following two propositions with opposite results.

Proposition 2-1

Assume: (1) the optimal final stock of wealth associated with the different prices of a house are the same and positive, and the stock of saving at the time of replacement associated with the different prices are positive; (2) the consumption and housing services are independent items in his utility function; (3) the consumer's time preference equals the market rate of interest; (4) the new house yields higher use-value than the old house, then the increase in the price of a house would not delay the replacement time, provided that (5) the final stock of wealth does not fall for the given replacement time: i. e.,

Assume (1) $W_T(P_1) = W_T(P_2) > 0$; $S_{t^*(P_1)}, S_{t^*(P_2)} > 0$

(2) $\frac{\partial^2}{\partial C_t \partial h_t} = 0$

(3) $r = \mu$

(4) $h' e^{-\rho(T-b+1-t^*(P_1))} \geq h^o e^{-\rho[t^*(P_1)+1-t_o^*]}$

(5) $\left.\frac{dW_T}{dP}\right|_{t^* \text{ fixed}} \geq 0$

then $P_1 < P_2$ implies $t^*(P_1) \geq t^*(P_2)$.

Proof:

(i) To show that the present value of utility would not fall, i.e., $\Psi[t^*(P_2), P_2] \geq \Psi[t^*(P_1), P_1]$ for $P_2 > P_1$

where $\Psi(t^*, P) = \sum_{t=1}^{T} U(C_t, h_t)(1+\mu)^{1-t} = V(W_T)(1+\mu)^{1-T}$

$$t^*(P_i) \ni \Psi[t^*(P_i), P_i] \geq \Psi[t^*, P_i] \; \forall t^*, \qquad i = 1, 2.$$

By assumption (1) and (5), and by Proposition 1 we have

$$\Psi[t^*(P_1), P_1] \leq \Psi[t^*(P_1), P_2].$$

Thus by definition of $t^*(P)$ we obtain:

$$\Psi[t^*(P_1), P_1] \leq \Psi[t^*(P_2), P_2] \tag{1.14}$$

Q.E.D.

(ii) To show $t^*(P_1) \geq t^*(P_2)$, assume the contrary, i.e., $t^*(P_1) < t^*(P_2)$. Assumptions (1), (2) and (3) ensure the optimal consumption sequence $\{C_t^1\}_{t=1}^{T}$ and $\{C_t^2\}_{t=1}^{T}$ are identical and constant over time, where C_t^j is the consumption at time t associated with P_j price.

Case 1 Assume $T = t^*(P_2) > t^*(P_1)$, let h_t^j be the use value of a house at time t associated with P_j price. Then $h_t^1 \geq h_t^2 \;\; \forall\, t=1,\cdots,T$. Thus $\Psi[t^*(P_1), P_1] > \Psi[t^*(P_2), P_2]$.

Case 2 Assume $T > t^*(P_2) = t^*(P_1) + b$ where b is any arbitrary positive integer such that the inequality above is satisfied.

We have

(1) $h_t^1 = h_t^2 \qquad t = 1, \cdots, t^*(P_1)$ (area A in Fig. 1)

(2) $h_2^1 = h_{t+b}^2 \qquad t = t^*(P_1)+1, \cdots, T-b$ (area B in Fig. 1)

(3) $h_{T-b+1}^1 \geq h_{t^*(P_1)+1}^2$ by assumption (4), which implies that

$$h_{t+T-b-t^*(P_1)}^1 \geq h_t^2 \qquad t = t^*(P_1)+1, \cdots, t^*(P_1)+b$$

(area C in Fig. 1)

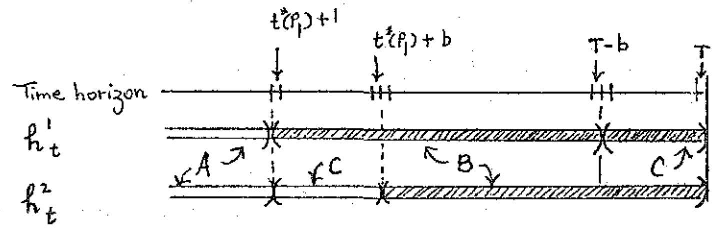

FIGURE 1

(Shaded area corresponds to the use value from the new house)

Thus we obtain

$$\sum_{t=t^*(P_1)+1}^{T} U(C_t^1, h_t^1)(1+\mu)^{1-t} > \sum_{t=t^*(P_1)+1}^{t^*(P_1)+b} U(C_t^1, h_{t+T-b-t^*(P_1)}^1)(1+\mu)^{1-t}$$

$$+ \sum_{t=t^*(P_1)+b+1}^{T} U(C_t^1, h_{t-b}^1)(1+\mu)^{1-t} \geq \sum_{t=t^*(P_1)+1}^{T} U(C_t^2, h_t^2)(1+\mu)^{1-t}.$$

Hence $\Psi[t^*(P_1), P_1] > \Psi[t^*(P_2), P_2]$, contradicting (1.14) above. Thus we conclude $t^*(P_1) \geq t^*(P_2)$.

Q.E.D.

Proposition 2-2

Under the identical assumptions (1) through (4) in Proposition 2-1, the increase in the price would not hasten the replacement time, provided that (5) the final stock of wealth declines for any given t^*, i.e.,

Assume (1) $W_T(P_1) = W_T(P_2) > 0; \quad S_{t^*(P_1)}, S_{t^*(P_2)} > 0$

(2) $\dfrac{\partial^2 U}{\partial C_t \partial h_t} = 0$

(3) $r = \mu$

(4) $h^1 e^{-e[T-b+1-t^*(P_2)]} \geq h^o e^{-e[t^*(P_2)+1-t_o^*]}$

(5) $\left.\dfrac{dW_T}{dP}\right|_{t^* \text{ fixed}} < 0$

then $P_1 < P_2$ implies $t^*(P_1) \leq t^*(P_2)$.

Proof. Similar to Proposition 2-1, use Corollary 1 instead of Proposition 1.

<5.2> Secondly we consider the effect of change in the rate of interest on the optimal consumption sequence $\{C_t\}_{t=1}^T$ and the optimal replacement timing t^*:

From (1.10) above we derive

(1.15) $\dfrac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}} = \left(\dfrac{1+r}{1+\mu}\right)$ for $S_{t^*}, W_T > 0$ and $\forall t = 1, \cdots, T-1$.

Differentiate (1.15) above with respect to r to obtain

(1.16) $a_1 \dfrac{dC_1}{dr} < a_1 a_2 \dfrac{dC_2}{dr} < \cdots < \prod_{i=1}^{t} a_i \dfrac{dC_t}{dr} < \cdots < \prod_{i=1}^{T} a_i \dfrac{dC_T}{dr}$

where $a_t = \left(\dfrac{1+r}{1+\mu}\right)\left(\dfrac{\partial^2 U/\partial C_t^2}{\partial^2 U/\partial C_{t-1}^2}\right) > 0 \qquad t = 2, 3\cdots, T; \; a_1 \equiv 1.$

Thus it is clear that if the optimal consumption in the first period rises as the interest rate rises, then the consumption in each following period will also rise. On the other hand if the optimal consumption in the final period falls as the interest rises, then so does the consumption in each preceding period. Thus we obtain Proposition 3 and its Corollary.

Proposition 3.

For given t^* assume $S_{t^*}, W_T > 0$, the $\dfrac{dC_1}{dr} \geq 0$ implies

(i) $\dfrac{dC_t}{dr} > 0 \qquad t = 2, \cdots, T$

(ii) $\dfrac{d\Psi}{dr} > 0.$

P r o o f.

(i) $\frac{dC_t}{dr} > 0 \qquad t = 2,\cdots,T$ is immediate by the equation (1.16).

(ii) To show $\frac{d\Psi}{dr} > 0$.

Differentiate (1.10) above with respect to r

$$(1.17)\quad \frac{\partial^2 U}{\partial C_t^2}\ \frac{dC_t}{dr} = V''\ \frac{dW_T}{dr}\left(\frac{1+r}{1+\mu}\right)^{T-t} + (T-t)\ V'\left(\frac{1+r}{1+\mu}\right)^{T-t-1}\left(\frac{1}{1+\mu}\right)\ \forall t$$

By the result above we have $\frac{dC_t}{dr} > 0 \quad t = 2,\cdots,T$, thus

$$(1.18)\quad V''\ \frac{dW_T}{dr}\left(\frac{1+r}{1+\mu}\right)^{T-t} + (T-t)\ V'\left(\frac{1+r}{1+\mu}\right)^{T-t-1}\left(\frac{1}{1+\mu}\right) < 0 \qquad t=2,\cdots,T$$

which implies $\frac{dW_T}{dr} > 0$. Differentiate (1.1) with respect to r to obtain

$$(1.19)\quad \frac{d\Psi}{dr} = \sum_{t=1}^{T} \frac{\partial U}{\partial C_t}\ \frac{dC_t}{dr}\ (1+\mu)^{1-t} + V'\ \frac{dW_T}{dr}\ (1+\mu)^{1-T}$$

Thus we have $\frac{d\Psi}{dr} > 0$.

Q. E. D.

Corollary 3

For given t^* assume S_{t^*}, $W_T > 0$, then $\frac{dC_T}{dr} \leq 0$ implies

(i) $\frac{dC_t}{dr} < 0 \qquad t = 1,\cdots,T-1$ (ii) $\frac{d\Psi}{dr} < 0$,

As in the previous section <5.1> the changes in the final stock of wealth has a significant implication to the changes in the consumption sequence and the present value of utility.

Proposition 4.

For given t^* assume S_{t^*}, $W_T > 0$, then $\frac{dW_T}{dr} \leq 0$ implies

(i) $\frac{dC_t}{dr} < 0 \qquad t = 1,2,\cdots,T-1. \qquad \frac{dC_T}{dr} \leq 0$,

(ii) $\frac{d\Psi}{dr} < 0$.

P r o o f.

Recall

$$(1.17)\quad \frac{\partial^2 U}{\partial C_t^2}\ \frac{dC_t}{dr} = V''\ \frac{dW_T}{dr}\left(\frac{1+r}{1+\mu}\right)^{T-t} + (T-t)\ V'\left(\frac{1+r}{1+\mu}\right)^{T-t-1}\left(\frac{1}{1+\mu}\right)\ \forall t$$

Thus $\frac{dW_T}{dr} \leq 0$ implies $\frac{\partial^2 U}{\partial C_t^2}\ \frac{dC_t}{dr} > 0 \quad t = 1,\cdots,T-1$ and $\frac{\partial^2 U}{\partial C_T^2}\ \frac{dC_T}{dr} \geq 0$, hence $\frac{dC_t}{dr} < 0 \quad t = 1,\cdots,T-1$ and $\frac{dC_T}{dr} \leq 0$. By the equation (1.19) above, we have $\frac{d\Psi}{dr} < 0$.

Q. E. D.

Now the question we would like to ask is what factors are involved in the determination of the sign[25] of dC_1/dr and dC_T/dr. Expressing the equation (1.17) explicitly we have,

$$(1.20)\quad U_{tt}\frac{dC_t}{dr} = a_t\frac{dW_T}{dr} + a_t(T-t)b$$

$$= a_tB - a_t\sum_{t=1}^{T}(1+r)^{T-t}\frac{dC_t}{dr} + a_t(T-t)b \quad \forall t$$

where (a) $U_{tt} = \dfrac{\partial^2 U}{\partial C_t^2} < 0, \quad a_t = V''\left(\dfrac{1+r}{1+\mu}\right)^{T-t} < 0$

(b) $b = V'/V''(1+r)$

(c) $$B = T(1+r)^{T-1}(S_o - D_o^o) + \sum_{t=1}^{T}(T-t)(1+r)^{T-t-1}s_t$$

$$+ (T-t^*)(1+r)^{T-t^*-1}(H^o_{t^*} - P) + \bar{d}\sum_{t=1}^{t^*}(T-t^*-1+t)$$

$$(1+r)^{T-t^*-2+t} + f\sum_{t=t^*+1}^{T}(t-t^*-1)(1+r)^{t-t^*-2}$$

(d) $$\frac{dW_T}{dr} = a_t\left(B - \sum_{t=1}^{T}(1+r)^{T-t}\frac{dC_t}{dr}\right)$$

Collecting terms and presenting them in a matrix form, we have $Ax = b$, or

$$(1.21)\quad \begin{bmatrix} U_{11}+a_1(1+r)^{T-1} & a_1(1+r)^{T-2} & \cdots & a_1 \\ a_2(1+r)^{T-1} & U_{22}+a_2(1+r)^{T-2} & \cdots & a_2 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_T(1+r)^{T-1} & a_T(1+r)^{T-2} & \cdots & U_{TT}+a_T \end{bmatrix} \begin{bmatrix} \frac{dC_1}{dr} \\ \frac{dC_2}{dr} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dC_T}{dr} \end{bmatrix} = \begin{bmatrix} a_1(B+(T-1)b) \\ a_2(B+(T-2)b) \\ \cdot \\ \cdot \\ a_{T-1}(B+b) \\ a_TB \end{bmatrix}$$

The determinant of A and the typical cofactor of A are given by[26]

$$(1.22)\quad |A| = \prod_{t=1}^{T}U_{tt} + \sum_{k=1}^{T}a_k(1+r)^{T-k}\prod_{t\neq k}U_{tt}$$

where $\prod_{t=1}^{T}U_{tt} = U_{11}\cdot U_{22}\cdot \cdots \cdot U_{TT}$

$$(1.23)\quad A_{ij} = -a_j(1+r)^{T-i}\prod_{t\neq i,j}U_{tt} \quad i\neq j, \quad i,j=1,2,\cdots,T$$

$$A_{ii} = \prod_{t\neq i}U_{tt} + \sum_{k\neq i}a_k(1+r)^{T-k}\prod_{t\neq i,k}U_{tt} \quad i=1,2,\cdots,T$$

Solving for dC_j/dr we have

$$(1.24)\quad \frac{dC_j}{dr} = \frac{a_j}{|A|}\{B\prod_{t\neq j}U_{tt} + b(T-j)\prod_{t\neq j}U_{tt} + b\sum_{i\neq j}a_i(1+r)^{T-i}\prod_{t\neq i,j}U_{tt}(i-j)\}$$

$j=1,2,\cdots,T.$

Thus

$$(1.25) \quad \frac{dC_1}{dr} = \frac{a_1}{|A|} \{B \prod_{t \neq 1} U_{tt} + b(T-1) \prod_{t \neq 1} U_{tt} + b \sum_{i \neq 1} a_i (1+r)^{T-i} \prod_{t \neq i,1} U_{tt}(i-1)\}$$

It is clear that the necessary condition for dC_1/dr to be non-negative is to have $B > 0$. Recalling the equation (1.20-c) on B, we note the difficulty in signing B due to the unknown sign pattern of saving (s_t). It can be said, however, that the increase in the initial stock of saving S_o or the increase in the market value of the old house $H^o_{t^*}$, or the decrease in the initial debt D^o_o will have a tendency to increase B and hence the consumption. On the other hand the necessary condition for dC_T/dr to be non-positive is to have $B < 0$ as can be seen by choosing $j = T$ in the equation (1.24).

Concerning the relationship between the interest rate and the replacement time we introduce the following proposition, which states that if the debt payment period is relatively long and the final outstanding debt is exogenously given constant, then the increase in interest rate will hasten the replacement,[27] i.e.,

Proposition 5

Assume (1) $\bar{t}$ to be such that $(t^* + \bar{t} - T)\ln(1+r) \geq 1$

(2) $D^1_T(r_1) = D^2_T(r_2) = \overline{D}$

then $r_1 < r_2$ implies $t^*(r_1) > t^*(r_2)$

Proof.

(i) To show $\left.\frac{dD^1_T}{dr}\right|_{t^* \text{ fixed}} > 0$.

Differentiate equation (1.4b') with respect to r to obtain

$$(1.26) \quad \frac{dD^1_T}{dr} = (P - \phi)\left\{\frac{1}{(1+r)^{\bar{t}+1} - (1+r)}\right\} \xi(t^*)$$

where

$$\xi(t^*) = (t^* + \bar{t} - T)(1+r)^{T-t^*} - B(t^*)\bar{t}$$

$$B(t^*) = \frac{1 - (1+r)^{T-t^*-\bar{t}}}{1 - (1+r) - t^*}$$

Since $\xi(T) = 0$ and by the assumption (1) $\frac{d\xi}{dt^*} < 0$, we have $\xi(t^*) > 0$ $t^* \varepsilon [1, T]$, which implies $\frac{dD^1_T}{dr} > 0$.

(ii) By differentiating D^1_T with respect to t^* we have

$$(1.27) \quad \frac{dD^1_T}{dt^*} = (P - \phi)\frac{dB}{dt^*} > 0.$$

Thus by the constraint of assumption (2) we must have

$t^*(r_1) > t^*(r_2)$ for $r_1 < r_2$.

Q.E.D.

6. Concluding Remarks

We formulated a model in which the consumer maximizes the discounted present value of utility with respect to the consumption time path and replacement timing. We note that the derived optimal time path crucially depends upon the ratio of the interest rate to the subjective time preference. We also note such an optimal time path will shift either upward or downward depending upon the magnitude of the "wealth effect" as the price of a new house changes. The ambiguity involved here seems to be a typical one associated with any utility analysis.

Optimal replacement time is characterized in terms of its sensitivity to the changes in parameters such as the price of a new house and rate of interest. It is shown that one cannot unequivocally determine the direction of change in the replacement timing as the parameters change. The "wealth effect", debt constraints, and "financial position" must all be taken into account.

Even though the analysis here is confined to the situation where replacement takes place exactly once, the model can be easily generalized to handle n replacements ($n = 0, 1, \cdots, T$); however only proposition 1 and its corollary can be established for the general model.

The author believes that further study concerning the optimal number of replacements and the changes in the parameters (price, interest rate) will yield fruitful return in the study of consumer durables.

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Footnotes

1. Some studies have been made concerning the optimal life of consumer durables (A.H. Fox [11], A.A. Alchian [2]). But alas, those studies are not dealt with in the proper framework of inter-temporal utility maximization.

2. The relevance of such study to the demand for consumer durables has been neglected in the past. However such study will certainly shed some light on the hitherto obscure relationship between the components of the demand for durables.

3. In contrast to the firm and organization, probably it is more appropriate to assume a finite, fixed time horizon for the consumer theory. Expected remaining life length may constitute the upper bound for the horizon.

4. See G.A.D. Preinriech [21], p. 44 for the discussion of H. Hotelling and Preinreich.

5. The quasi-rent function given in this form, $Q(L_j, a)$ can take into account technological improvement, and obsolescence factor.

6. If we assume the scrap value changes with the time of scrapping, i.e., $S = S(T_j)$, then the expression (4) will become $Q(T_j) = r(S + G_{j-1}) - S'(T_j)$, which results in earlier scrapping under the usual assumptions of $Q'(T_j) < 0$, $S'(T_j) < 0$.

7. Note: G_n here corresponds to the "goodwill" expression G in equation (2) above.

8. The sequence of the life of a machine $\{T_j\}_{j=1}^{n}$ is monotonically decreasing until it reaches the limit $\tilde{T} \geq 0$. The sequence of the "goodwill" $\{G_j\}_{j=1}^{n}$ is monotonically increasing until it reaches the limit $\tilde{G}$, such $\tilde{G}$ is associated with the limit $\tilde{T}$. If $\tilde{T}$ is zero, then the sum of the infinitely extended sequence may be finite, i.e., $\sum_{j=1}^{\infty} T_j = \hat{T} < \infty$, in such a case, a project involving the time horizon longer than $\hat{T}$ may find the machine concerned is not appropriate.

9. This, of course, does not necessarily follow if the time horizon is fixed.

10. See the criticism by Preinreich [21] of this formulation.

11. By assuming the rate of production to be unit constant, i.e., $x(t) = 1$, equation (8) yields something similar to Smith's [23] constant annual cost stream. Additional assumption on the interest rate and the scrap value in the nature of both being insignificant, i.e., $r = S = 0$, will reduce equation (8) into the objective function used by Brems [4], Fox [11], Clapham [5], i.e., $w = \frac{C}{T} + \frac{1}{T}\int_0^T M(t)\,dt$. By assuming operating, maintenance cost to be linear in the age of the machine, i.e., $M(t) = at$, we will have the so-called "square-root formula" $T = \sqrt{\frac{2C}{a}}$.

12. Naturally the same expression can be obtained by maximizing

$$G = \sum_{k=o}^{\infty} e^{-rkT}\left\{\int_o^T Q(t)e^{-rt}\,dt + Se^{-rT} - C\right\}.$$ See Lutz and Lutz [17], Alchian [2] and Smith [22].

13. This problem is first posed by Alchian [2] and later by S. Dreyfus [6]. Dreyfus uses the functional equation technique of dynamic programming. The quotation is from Alchian [2], p. 12.

14. Proof goes as follows: Let G(a, b) be the maximum goodwill associated with b number of machines and time horizon a. Then we have $G[\overline{T}, n(\overline{T})] \geq G[T(\overline{n}), \overline{n}]$ for $\overline{T} \geq T(\overline{n})$. We want to show $n(\overline{T}) \geq \overline{n}$. Assume the contrary, i.e., $n(\overline{T}) = n_1 < \overline{n}$. Then $G[\overline{T}, n(\overline{T})] = G[T(n_1), n_1] < G[T(\overline{n}), \overline{n}]$, where $T(n_1) = \sum_{i=1}^{n_1} T_i < T(\overline{n})$. Thus contradiction is obtained and we conclude $n(\overline{T}) \geq \overline{n}$.

15. If $T \geq T(\overline{n}) \Longrightarrow n(\overline{T}) = \overline{n}$, then the optimal replacement timing is exactly the same as the finite chain case. An interesting case is $\overline{T} \geq T(\overline{n})$ implies $n(\overline{T}) > \overline{n}$. In particular if $\overline{T} = T(\overline{n})$, then it can be shown that $T_1 > m_i \; \forall i=1,\cdots,n(\overline{T})-1 \quad \cdots \quad T_{\overline{n}} > m_i \; \forall i=1,\cdots, n(\overline{T}) - \overline{n}+1$, where m_i is the optimal life of ith machine in the fixed horizon case.

16. If $\overline{T} \geq T(\overline{n})$, then we assume $n(\overline{T})$ to be strictly larger than $\overline{n}$. Otherwise the optimal policy would be identical to the finite case, except when $\overline{T} > T(\overline{n})$ and $n(\overline{T}) = \overline{n}$. In such a case the entrepreneur will have $\overline{T} - T(\overline{n})$ of "doing-nothing" period.

17. Rewriting the equation (15) we have

$$(15') \qquad Q(m_j) = A + \int_o^{m_{j+1}} Q'(t)e^{-rt}\,dt$$

$$A = r(S - C) + Q(0) \qquad\qquad j=1,\cdots,k-1$$

Thus given three possible cases (1) $m_k = m_{k-1}$, (2) $m_k > m_{k-1}$, (3) $m_k < m_{k-1}$, we have three forms of the sequence.

18. Note the equation (12) can be written as

$$Q(T) = \frac{r}{1 - e^{-rT}}\{\int_o^T Q(t)e^{-rt}\,dt + S - C\}.$$

19. If the consumer owns a house completely then $\overline{d}$ and the initial outstanding debt, D_o^o, are both equal to zero.

20. We assume that the date $(t^* + \overline{t})$ when the consumer completes his debt payment is farther than the time horizon T. Note $\overline{t}$ is given by

$$(1 + r)^{\overline{t}}\{P - \phi(P)\} - f(P) \sum_{t=t^*+1}^{t^*+\overline{t}} (1 + r)^{t-t^*-1} = 0.$$

Thus f(P) can be expressed by the following

$$f(P) = \{P - \phi(P)\}\{\frac{r}{1 - (1+r)^{-\overline{t}}}\}.$$

Substitution of the above into (1.4b) yields (1.4b').

21. We would only consider the case where $C_t > 0 \; \forall t$. This is possible by assumption (1.8b) $(\partial/\partial C_t)U(0, h_t) = \infty$.